Question 1 **Start a new page**

Marks

(a) (i) The base of a certain solid S_1 is the region bounded by the parabola $y^2 = 4ax$ and the line $x = a$, where $a > 0$. **3**

By taking slices parallel to the y-axis in this base where each cross-section is an equilateral triangle, find the volume of S_1.

(ii) The area bounded by $y^2 = 4ax$ and the line $x = a$ is rotated about the line $x = a$ to form a solid of revolution.

By considering slices parallel to the x–axis:

(α) Show that the cross-sectional area A is given by: **2**

$$A = \pi\left(a^2 - \frac{1}{2}y^2 + \frac{1}{16a^2}y^4\right).$$

(β) Hence, find the volume of the solid of revolution. **2**

(b) A particle P of mass m kg, is attached to the end of a light wire 5 cm long which rotates as a conical pendulum with uniform speed in a horizontal plane below a fixed point O to which the wire is attached. The particle rotates so that the angular velocity is ω rads/sec.

(i) Show that the angular velocity is $\frac{26\pi}{5}$ rads/sec when the particle is rotating at 156 rpm. **1**

(ii) Find the semi-vertical angle θ of the conical pendulum (answer to the nearest degree and take $g = 9.8$ m/s^2). **2**

(c) A particle moves in a straight line. It is placed at the origin O on the x-axis and is then released from rest.

When it is at position x, the acceleration $\ddot{x}$, of the particle is given by:

$$\ddot{x} = -9x + \frac{5}{(2-x)^2}.$$

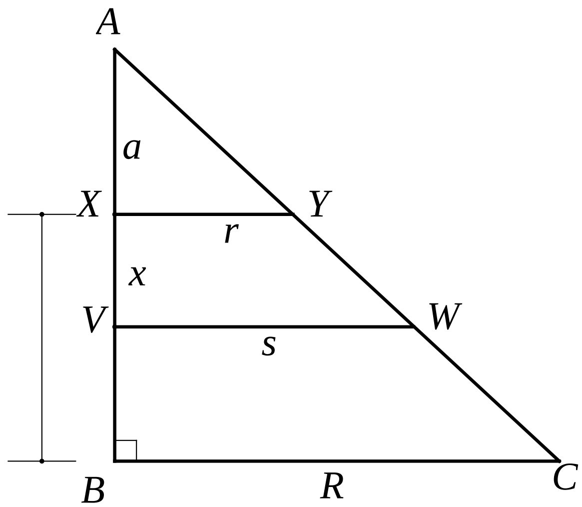
(i) Show that: $v^2 = \dfrac{x(3x-5)(3x-1)}{2-x}$ for $x \neq 2$. **3**

(ii) Prove that the particle moves between two points on the x-axis, and find these points. **2**

Question 2 **Start a new page**

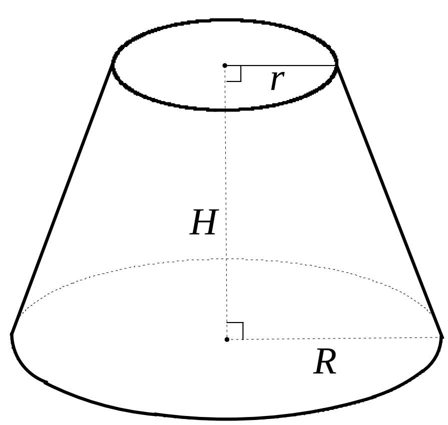
Marks

(a) Consider the right-triangle ABC, where XY and BC are the lengths r and R respectively. Given VW is parallel to XY and BC. The distance between XY and BC is H, the length $VW = s$, $XV = x$ and length $AX = a$, as shown.



(i) Show using similar triangles: $s = \dfrac{R-r}{H}\left(x + \dfrac{Hr}{R-r}\right)$. **3**

(ii) The frustum of a cone where the radius of the top and bottom faces are r and R respectively and the height is H, is shown below.



By considering a cross-sectional slice of the frustum parallel to the top face at a distance x units from the top, and using **integration,** show that the volume V of the frustum is given by : **3**

$$V = \frac{\pi H}{3}\left(R^2 + Rr + r^2\right) .$$

Question 2 Continued

Marks

(b) An object of unit mass falls under gravity through a resistive medium.
The object falls from rest from a height of 50 metres above the ground.
The resistive force, in Newtons, is of magnitude $\frac{1}{100}$ the square of the objects speed v ms^{-1} when it has travelled a distance x metres. Let g be the acceleration due to gravity in ms^{-2}.

(i) Draw a diagram to show the forces acting on the body. **1**
Hence, show that the equation of motion of the body is:

$$\ddot{x} = g - \frac{v^2}{100}.$$

(ii) Show that the terminal speed, u ms^{-1}, of the body is given by: **1**

$$u = \sqrt{100g}.$$

(iii) Prove that: $\ddot{x} = v\frac{dv}{dx}$. **1**

(iv) Show that: $\frac{v^2}{u^2} = 1 - e^{-\frac{x}{50}}$. **3**

(v) Find the distance fallen when the object has reached a speed equal to 50% of its terminal speed (correct to 1 decimal place). **2**

(vi) Find the speed attained, as a percentage of the terminal speed, when the object hits the ground (correct to 1 decimal place). **1**

<u>Question 3</u> **Start a new page**

Marks

(a) The region bounded by the curves $y = \dfrac{1}{x+1}$ and $y = \dfrac{1}{x+2}$ and the lines $x = 0$ and $x = 2$, is rotated about the y-axis, forming a solid of revolution with a volume of V units^3.

(i) Show that: $V = 2\pi \displaystyle\int_0^2 \frac{x}{(x+1)(x+2)}\,dx$. **2**

(ii) Find V, correct to three significant figures. **3**

(b) A vehicle is travelling along a horizontal straight road with a speed of 42 ms^{-1}. The engine is stopped as it passes a point marked O on the road and then the car is allowed to come to rest at a point B. The frictional resistance force is $\frac{1}{7}$ of the weight of the car and the air resistive force is $\dfrac{v}{14}$ per unit mass, where v is the speed of the car.

(i) If x is the distance travelled in metres, explain why $\ddot{x} = -\left(\dfrac{v+2g}{14}\right)$, **1**

where g is the acceleration due to gravity, in ms^{-2}.

(ii) Find the distance travelled (to the nearest metre) **and** the exact time taken for the car to come to rest once the engine is stopped. **4**

Take g = 10 ms^{-2}.

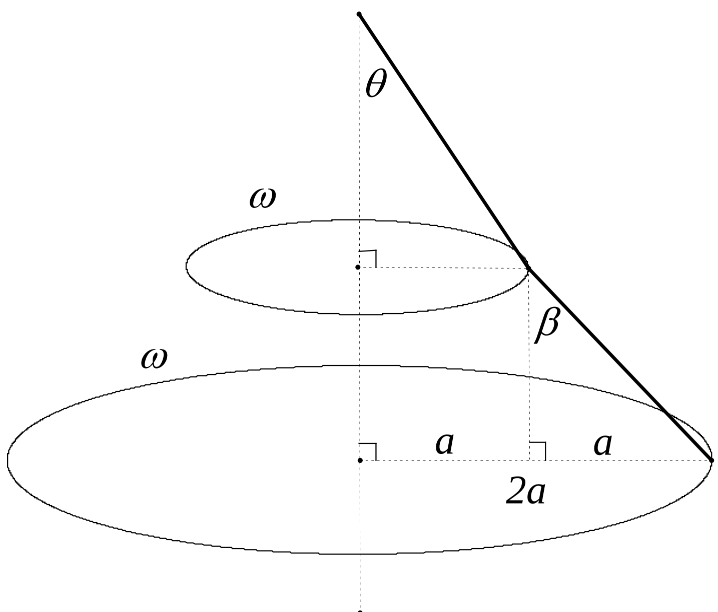
Question3 Continued

(c) A light inextensible string ABC is such that $AB = \frac{5a}{3}$ and $BC = \frac{5a}{4}$.

A particle of mass m kg is attached to the string at C and another particle of mass $7m$ kg is fixed at B. The end A is tied to a fixed point and the whole system rotates steadily about the vertical AH (as shown), in such a way that B and C describe horizontal circles of radii a and $2a$ respectively and each has the same angular velocity ω.

(i) By resolving the forces at C, **2**

show that the tension in the string BC is $\frac{5mg}{3}$ Newtons.



(ii) Hence, find the tension in the part of string AB. **1**

(iii) Find the speed of the particle at B. **2**

<u>Question 4</u> **Start a new page**

Marks

(a) A rectangular hyperbola has the equation $x^{\blacksquare} - y^{\blacksquare} = 8$.

Write down its eccentricity, the coordinates of the foci and the equation of each directrix. **5**

Sketch the curve, indicating on your diagram each focus, directrix and asymptote.

(b) This curve is rotated anti-clockwise through 45^0, where the equation of the curve takes the form $xy = 4$.

(i) Prove that the equation of the normal to the rectangular hyperbola $xy = 4$ at the point $P\left(2p, \dfrac{2}{p}\right)$ is $py - p^3x = 2(1 - p^4)$. **2**

(ii) If this normal meets the hyperbola again at $Q\left(2q, \dfrac{2}{q}\right)$ with parameter q, prove that $q = -\dfrac{1}{p^3}$. **2**

(iii) Hence, or otherwise, explain why there exists only one chord of the hyperbola where the gradients of the normal, at both ends, are equal.

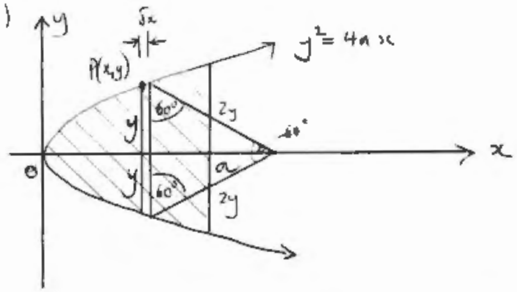
Find the equation of this special chord PQ. **3**

(iv) Find the equation of the locus of the midpoint R of the chord PQ, as p and q vary. **3**

<u>End of Exam Paper</u>

SOLUTIONS TO: YEAR 12 – Term 2 – ME 2 – 2008

Qu. (1) (a) (i)



Area of equilateral $\Delta = \frac{1}{2} \times 2y \times 2y \times \sin 60$

$= 2y^2 \times \frac{\sqrt{3}}{2}$

$= \sqrt{3}y^2$

$= \sqrt{3} \times 4ax \qquad (y^2 = 4ax)$

$\therefore A(x) = 4\sqrt{3}ax$ (1)

Volume of cross-sectional slice $= \delta V = 4\sqrt{3}ax\,\delta x$

Volume of Solid $= S_1 = \lim_{\delta x \to 0} \sum_{x=0}^{x=a} 4\sqrt{3}ax\,\delta x$

$= \int_0^a 4\sqrt{3}ax\,dx$ (1)

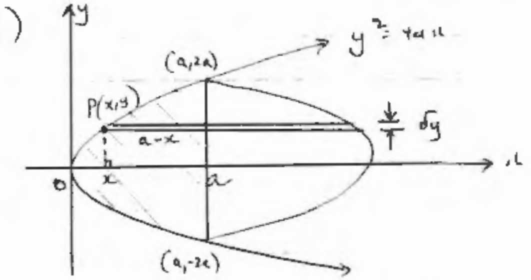
$= 4\sqrt{3}a\left[\frac{x^2}{2}\right]_0^a$

$= 4\sqrt{3}a\left(\frac{a^2}{2} - 0\right)$

$= \underline{2\sqrt{3}a^3}$ units^3 (1)

[3]

(ii)



(α) A = Area of cross-section $= \pi r^2$

$= \pi(a-x)^2$ (1)

$= \pi\left(a - \frac{y^2}{4a}\right)^2 \qquad$ (where $y^2 = 4ax$)

$= \pi\left(a^2 - 2.a.\frac{y^2}{4a} + \left(\frac{y^2}{4a}\right)^2\right)$ (1)

$\therefore A = \pi\left(a^2 - \frac{1}{2}y^2 + \frac{1}{16a^2}y^4\right)$

[2]

(β) Volume of cross-sectional disc $= \pi\left(a^2 - \frac{1}{2}y^2 + \frac{1}{16a^2}y^4\right)\delta y$

$V = \pi\int_{-2a}^{2a}\left(a^2 - \frac{1}{2}y^2 + \frac{1}{16a^2}y^4\right)dy$

$= 2\pi\int_0^{2a}\left(a^2 - \frac{1}{2}y^2 + \frac{1}{16a^2}y^4\right)dy$ (1)

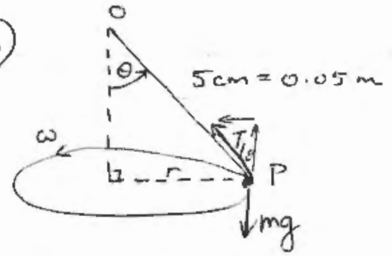
$= 2\pi\left[a^2y - \frac{1}{6}y^3 + \frac{y^5}{80a^2}\right]_0^{2a}$

$= 2\pi\left(2a^3 - \frac{1}{6}\times 8a^3 + \frac{32a^5}{80a^2} - 0\right)$

$= \frac{32}{15}\pi a^3$ (1)

[2]

(1) (b)



(i) $\omega = 156$ revs/min

$= 156 \times 2\pi$ radians/min

$= \frac{156 \times 2\pi}{60}$ radians/sec (1)

$\omega = \frac{26\pi}{5}$ radians/sec

[1]

(ii) Resolving forces at P:

Vertically: $0 = mg - T_1\cos\theta$

$mg = T_1\cos\theta$ ---- (1)

Normally: $mr\omega^2 = T_1\sin\theta$ --- (2)

but $\sin\theta = \frac{r}{0.05} \implies r = 0.05\sin\theta$ --- (3)

(2) ÷ (1): $\frac{T\sin\theta}{T\cos\theta} = \frac{mr\omega^2}{mg}$

$\tan\theta = \frac{r\omega^2}{g}$

$\therefore \tan\theta = \frac{0.05\sin\theta}{9.8} \times \frac{26^2\pi^2}{5^2}$ (1) (From (3) and $\omega = \frac{26\pi}{5}$)

$\sec\theta = \frac{0.05 \times 26^2 \times \pi^2}{9.8 \times 5^2}$

or $\cos\theta = \frac{9.8 \times 5^2}{0.05 \times 26^2 \times \pi^2}$

$\underline{\theta = 43°}$ (nearest degree) (1)

[2]

1.(c) (i) $\ddot{x} = -9x + \dfrac{5}{(2-x)^2}$

$$\frac{d}{dx}\left(\tfrac{1}{2}v^2\right) = -9x + 5(2-x)^{-2}$$

$$\tfrac{1}{2}v^2 = -\frac{9x^2}{2} + 5(2-x)^{-1} + c \quad ①$$

When $x=0, v=0$:

$$0 = 5(2)^{-1} + c$$

$$c = -\frac{5}{2}.$$

$$\tfrac{1}{2}v^2 = \frac{5}{2-x} - \frac{9x^2}{2} - \frac{5}{2}$$

$$v^2 = \frac{10}{2-x} - 9x^2 - 5 \quad ①$$

$$= \frac{10 - 9x^2(2-x) - 5(2-x)}{2-x}$$

$$= \frac{10 - 18x^2 + 9x^3 - 10 + 5x}{2-x} \quad ①$$

$$= \frac{x(9x^2 - 18x + 5)}{2-x}$$

$$\therefore v^2 = \frac{x(3x-1)(3x-5)}{2-x}$$

[3]

(ii) Particle is at rest $(v=0)$:

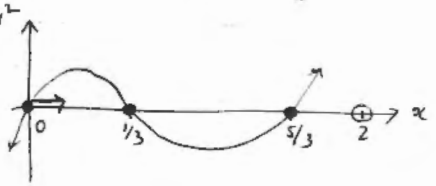
$$0 = \frac{x(3x-1)(3x-5)}{2-x}$$

$$0 = x(3x-1)(3x-5), \quad x \neq 2$$

$$x = 0,\ x = \tfrac{1}{3},\ x = \tfrac{5}{3}$$

at $x=0$, $\ddot{x} = \frac{5}{4}$

$\therefore$ particle moving right.



$v^2 \geq 0$ when $0 \leq x \leq \frac{1}{3}$ and $x \geq \frac{5}{3}, x \neq 2$ ①

However, since particle was at $x=0$ initially, then the particle

Qu 2

(a) (i) $\dfrac{a}{a+H} = \dfrac{r}{R}$ (corresponding sides of similar Δ's are in proportion) ①

$$aR = ar + Hr$$

$$a = \frac{rH}{R-r} \quad \text{---- ①} \qquad ①$$

Similarly: $\dfrac{a}{a+x} = \dfrac{r}{s}$

$$as = r(a+x)$$

$$s = \frac{r}{a}(a+x) \quad \text{---- ②} \qquad ①$$

Sub. ① into ②: $s = r \times \dfrac{(R-r)}{rH}\left(\dfrac{rH}{R-r} + x\right)$

$$\therefore s = \frac{R-r}{H}\left(\frac{rH}{R-r} + x\right) \text{ as required}$$

[3]

(ii) Volume of cross-sectional slice $= \pi s^2\, \delta x$

$$= \pi\frac{(R-r)^2}{H^2}\left(\frac{rH}{R-r} + x\right)^2 \delta x$$

Volume of solid $= V = \dfrac{\pi(R-r)^2}{H^2}\displaystyle\int_0^H \left(\frac{rH}{R-r} + x\right)^2 dx$ ①

$$= \frac{\pi(R-r)^2}{H^2}\int_0^H \left(\frac{r^2H^2}{(R-r)^2} + \frac{2xrH}{R-r} + x^2\right) dx$$

$$= \frac{\pi(R-r)^2}{H^2}\left[\frac{r^2H^2x}{(R-r)^2} + \frac{x^2rH}{R-r} + \frac{x^3}{3}\right]_0^H$$

$$= \frac{\pi(R-r)^2}{H^2}\left(\frac{r^2H^3}{(R-r)^2} + \frac{H^3r}{R-r} + \frac{H^3}{3}\right) \quad ①$$

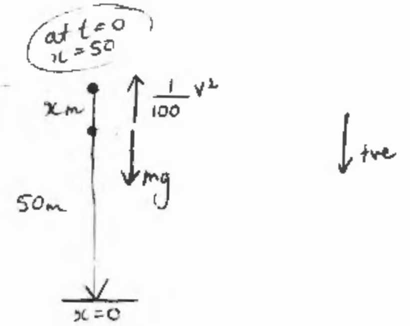
$$= \frac{\pi(R-r)^2}{H^2} \cdot \frac{H^3}{3(R-r)^2}\left(3r^2 + 3r(R-r) + (R-r)^2\right) \quad ①$$

$$= \frac{\pi H}{3}\left(3r^2 + 3rR - 3r^2 + R^2 - 2Rr + r^2\right)$$

$$= \frac{\pi H}{3}\left(R^2 + Rr + r^2\right)$$

[3]

(b)



(i) $m\ddot{x} = mg - \frac{1}{100}v^2$ ①

since $m = 1$

$\ddot{x} = g - \frac{v^2}{100}$ as required [1]

(ii) Terminal velocity ($\ddot{x} = 0$)

$0 = g - \frac{1}{100}u^2$ ①

$u^2 = 100g$

$u = \sqrt{100g}$ [1]

(iii) $\frac{dv}{dt} = \frac{dv}{dx} \cdot \frac{dx}{dt}$ ①

$\ddot{x} = \frac{dv}{dx} \cdot v$ as required. [1]

(iv) $v\frac{dv}{dx} = g - \frac{v^2}{100}$

$\frac{dv}{dx} = \frac{100g - v^2}{100v}$

$\frac{dx}{dv} = \frac{100v}{100g - v^2}$

$= 50\left(\frac{2v}{100g - v^2}\right)$

$x = -50\ln(100g - v^2) + c$ ①

When $x = 0, v = 0$, $100g = u^2 \Longrightarrow$

$0 = -50\ln u^2 + c$

$c = 50\ln u^2$.

$\therefore x = -50\ln(u^2 - v^2) + 50\ln u^2$ ①

$\frac{-x}{50} = \ln\left(\frac{u^2 - v^2}{u^2}\right)$

$\frac{-x}{50} = \ln\left(1 - \frac{v^2}{u^2}\right)$

$1 - \frac{v^2}{u^2} = e^{-\frac{x}{50}}$ ①

$\frac{v^2}{u^2} = 1 - e^{-\frac{x}{50}}$, as required. [3]

(v) $v = \frac{1}{2}u$

$\frac{v^2}{u^2} = \frac{1}{4}$

$1 - e^{-\frac{x}{50}} = \frac{1}{4}$ ①

$e^{-\frac{x}{50}} = \frac{3}{4}$

$-\frac{x}{50} = \ln\frac{3}{4}$

$x = -50\ln\frac{3}{4}$

$\doteqdot 14.38410362$

$\therefore \underline{x = 14.4 \text{ (1 dp)}}$ ① [2]

(vi) Body hits ground, when $x = 50$:

$\frac{v^2}{u^2} = 1 - e^{-1}$

$\frac{v}{u} = \sqrt{1 - e^{-1}}$

$= 0.795060097$

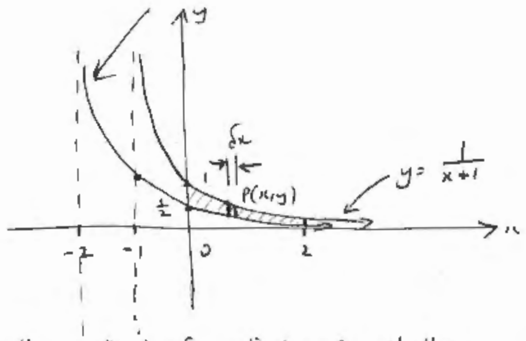
$\frac{v}{u} = 79.5\%$ (1 dp) ①

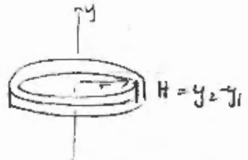
$v = 79.5\% \times u$

$\therefore$ Speed of particle when it hits the ground is 79.5% of terminal speed. [1]

Qu 3

(a)





(i) By the method of cylindrical shells:

Volume of cylindrical shell $= \delta V = 2\pi r H$

$= 2\pi x \left(\frac{1}{x+1} - \frac{1}{x+2} \right)$ ①

$= 2\pi x \left(\frac{x+2-(x+1)}{(x+1)(x+2)} \right)$

$= 2\pi x \times \frac{1}{(x+1)(x+2)}$. ①

$\delta V = \frac{2\pi x}{(x+1)(x+2)}$

Volume of solid $= V = 2\pi \int_0^2 \frac{x}{(x+1)(x+2)}$ [2]

(ii) Let $\frac{x}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$

$x = A(x+2) + B(x+1)$

Let $x=-1$: $-1 = A$
Let $x=-2$: $-2 = -B$, $B = 2$ } ①

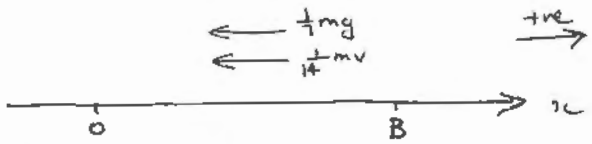
$\therefore V = 2\pi \int_0^2 \left(\frac{2}{x+2} - \frac{1}{x+1} \right) dx$

$= 2\pi \left[2\ln(x+2) - \ln(x+1) \right]_0^2$ ①

$= 2\pi \left(2\ln 4 - \ln 3 - 2\ln 2 + \ln 1 \right)$

$= 2\pi \left(2\ln 2 - \ln 3 \right) = 1.81$ (3 sf) u^3 ① [3]

(b)



(i) $m\ddot{x} = -\frac{mg}{7} - \frac{mv}{14}$ ①

$\ddot{x} = -\left(\frac{v}{14} + \frac{g}{7} \right)$

$\ddot{x} = -\left(\frac{v+2g}{14} \right)$ [1]

(ii) Take $g = 10$: $\ddot{x} = -\left(\frac{v+20}{14} \right)$

$\frac{dv}{dt} = -\left(\frac{v+20}{14} \right)$

$\frac{dt}{dv} = -\frac{14}{v+20}$

$t = -14 \int_{42}^{v} \frac{1}{v+20}\, dv$

$= -14 \left[\ln(v+20) \right]_{42}^{v}$

$= -14 \left[\ln(v+20) - \ln 62 \right]$

$t = 14 \ln \frac{62}{v+20}$ --- (A) ①

Time taken for particle to come to rest $(v=0)$

$t = 14 \ln \frac{62}{20}$

OR $t = 14 \ln 3.1$ ($\doteqdot 15.84$ sec) ①

From (A): $\frac{t}{14} = \ln \frac{62}{v+20}$

$\frac{62}{v+20} = e^{t/14}$

$62 e^{-t/14} = v + 20$

$v = 62e^{-t/14} - 20$ ①

$\frac{dx}{dt} = 62e^{-t/14} - 20$

$x = \int_0^{14\ln 3.1} \left(62e^{-t/14} - 20 \right) dt$

$= \left[-14 \times 62 e^{-t/14} - 20t \right]_0^{14\ln 3.1}$

$\therefore x = -868e^{-\ln 3.1} - 20 \times 14 \ln 3.1 + 14 \times 62$

$= -868 \times \frac{10}{31} - 280 \ln 3.1 + 868$

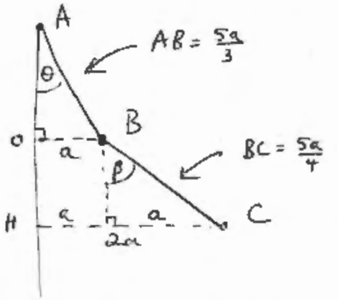
$= 271.2074088$

$x = \underline{271\text{ m (nearest metre)}}$ (1)

$\therefore$ It takes $\underline{14 \ln 3.1}$ sec and $\underline{271\text{ m}}$ for the car to completely come to rest.

[4]

(c)



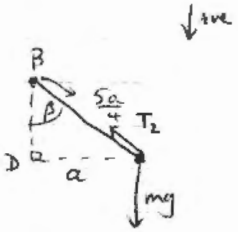
(i) At C, resolving forces:

<u>Vertically</u>: $0 = mg - T_2 \cos\beta$

$T_2 = \frac{mg}{\cos\beta}$

$= \frac{mg}{\frac{3}{5}}$

$T_2 = \frac{5mg}{3}$ (1)



$BD = \sqrt{\left(\frac{5a}{4}\right)^2 - a^2}$

$= \sqrt{\frac{25a^2}{16} - a^2}$

$BD = \frac{3a}{4}$

$\therefore \cos B = \frac{3a}{4} \div \frac{5a}{4}$

$\cos B = \frac{3}{5}$ (1)

[2]

(ii) Resolving forces <u>at B</u>:

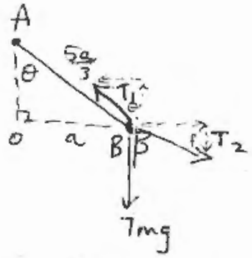
<u>Vertically</u>:

$0 = 7mg - T_1 \cos\theta + T_2 \cos\beta$

$-7mg = T_2 \cos\beta - T_1 \cos\theta$ --- (B)

$-7mg = \frac{5mg}{3} \times \frac{3}{5} - T_1 \times \frac{4}{5}$ (1)

$\underline{T_1 = 10mg}$



$AO = \sqrt{\left(\frac{5a}{3}\right)^2 - a^2}$

$= \sqrt{\frac{25a^2}{9} - a^2}$

$= \sqrt{\frac{16a^2}{9}}$

$AO = \frac{4a}{3}$

$\therefore \cos\theta = \frac{4a}{3} \div \frac{5a}{3}$

$\therefore \cos\theta = \frac{4}{5}$

(iii) <u>Need to find ω</u>:

At C: $T_2 \cos\beta = mg$ (Vertically)

$T_2 \sin\beta = 2am\omega^2$ (normally)

$\frac{T_2 \sin\beta}{T_2 \cos\beta} = \frac{2am\omega^2}{mg}$

$\tan\beta = \frac{2a\omega^2}{g}$

$\omega^2 = \frac{\tan\beta \times g}{2a}$

$= \frac{4/3\, g}{2a}$

$\omega = \sqrt{\frac{2g}{3a}}$ (1)

<u>At B</u>, $v = r\omega = a\sqrt{\frac{2g}{3a}} = \sqrt{\frac{2ag}{3}}$ (1)

[2]

Qu 4

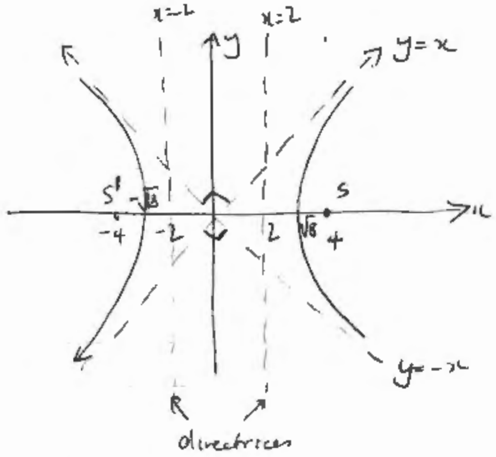
(a) $x^2 - y^2 = 8 \implies \frac{x^2}{8} - \frac{y^2}{8} = 1 \quad \therefore a = \sqrt{8},\ b = \sqrt{8}$

Using $b^2 = a^2(e^2-1)$ gives
$$8 = 8(e^2 - 1)$$
$$1 = e^2 - 1$$
$$\underline{e = \sqrt{2}} \quad (1)$$

Foci are S & S$'$, using $(\pm ae, 0) = (\pm 4, 0)$. (1)

Eq^{n}s of directrices are $x = \pm \frac{a}{e} \implies x = \pm \frac{\sqrt{8}}{\sqrt{2}}$

ie $\underline{x = \pm 2}$ (1)



Asymptotes $(\frac{1}{2})$
x-intercepts $(\frac{1}{2})$
directrices $(\frac{1}{2})$
Foci $(\frac{1}{2})$

[5]

(b) (i) $xy = 4 \implies y = 4x^{-1}$
$$\frac{dy}{dx} = -4x^{-2}$$
at $P(2p, \frac{2}{p})$, $\frac{dy}{dx} = \frac{-4}{(2p)^2} = \frac{-4}{4p^2} = \frac{-1}{p^2}$ (1)

(Alternative: $x\frac{dy}{dx} + y.1 = 0 \implies \frac{dy}{dx} = -\frac{y}{x}$
at $P(2p, \frac{2}{p})$, $\frac{dy}{dx} = -\frac{2/p}{2p} = \frac{-1}{p^2}$)

$\therefore$ Gradient of normal $= p^2$.

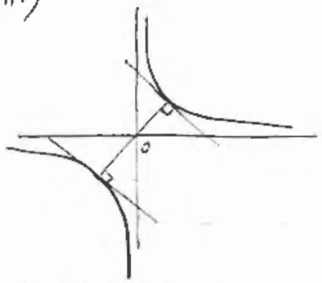
Eq^n of normal: $y - \frac{2}{p} = p^2(x - 2p)$ (1)
$$py - 2 = p^3(x - 2p)$$
$$py - p^3x = 2(1 - p^4)$$

[2]

(ii) If normal passes through $Q(2q, \frac{2}{q})$, then it satisfies equation:
$$p \times \frac{2}{q} - p^3 \times 2q = 2(1 - p^4)$$
$$2p - 2p^3q^2 = 2q - 2p^4q \quad (1)$$
$$p - q = p^3q^2 - p^4q$$
$$p - q = p^3q(q - p) \quad (1)$$
$$-1 = p^3q \quad (p \neq q)$$
$$\therefore q = \frac{-1}{p^3}$$

[2]

(iii)



From above, if PQ is a normal at P then $q = \frac{-1}{p^3}$ ie $p^3q = -1$.

also, PQ is a normal at Q then $p = -\frac{1}{q^3}$ ie $pq^3 = -1$

If PQ is a normal at both P and Q then $p^3q = pq^3$
ie $p^3q - pq^3 = 0$ ie $pq(p^2 - q^2) = 0$
ie $pq(p-q)(p+q) = 0$.

Since $p \neq 0, q \neq 0$ and $p \neq q$ then $\underline{p + q = 0}$ only.
ie $\underline{q = -p}$

Since $p^3q = -1$, then $p^3 \times -p = -1$
$$p^4 = 1$$
$$\underline{p = \pm 1}$$

If $p = 1$, Eq^n of normal is $1y - 1^3x = 2(1 - 1^4)$
ie $\underline{y = x}$

If $p = -1$, Eq^n of normal is $-y + x = 2(1 - 1)$
$\underline{y = x}$

Thus, there is only one chord of the hyperbola where where the gradients of the normal, at both ends, are equal. Its equation is $\underline{\underline{y = x}}$

(iv) Midpoint of PQ = R

$\therefore R = \left(\frac{2p+2q}{2}, \frac{2/p + 2/q}{2}\right)$

$R = \left(p+q, \frac{p+q}{pq}\right)$

$\therefore$ $x = p+q$ $\overset{\text{but}}{\Longrightarrow}$ $q = \frac{-1}{p^3}$

(1) $y = \frac{p+q}{pq}$ $pq = \frac{-1}{p^2}$ --- (A)

$\frac{x}{y} = \frac{p+q}{\frac{p+q}{pq}}$

$= pq$

$\frac{x}{y} = -\frac{1}{p^2}$ (from (A))

$y = -p^2 x$ OR $-\frac{y}{x} = p^2$ (1)

Since R lies on the normal, it satisfies eq^n.

ie $py - p^3x = 2(1-p^4)$

$y - p^2x = \frac{2}{p}(1-p^4)$

$-p^2x - p^2x = \frac{2}{p}(1-p^4)$

$-2p^2x = \frac{2}{p}(1-p^4)$

$-2x\left(\frac{-y}{x}\right) = \frac{2}{p}\left(1 - \frac{y^2}{x^2}\right)$

$2y = \frac{2}{p}\left(\frac{x^2-y^2}{x^2}\right)$

$4y^2 = \frac{4}{p^2}\left(\frac{x^2-y^2}{x^2}\right)^2$ (squaring)

$4y^2 = 4 \times -\frac{x}{y}\left(\frac{x^2-y^2}{x^2}\right)^2$ (1)

$y^3 = -\frac{(x^2-y^2)^2}{x^3}$

$x^3y^3 + (x^2-y^2)^2 = 0$

[3]